

Secants of Lagrangian Grassmannians

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Abstract

We study the dimensions of secant varieties of Grassmannian of Lagrangian subspaces in a symplectic vector space. We calculate these dimensions for third and fourth secant varieties. Our result is obtained by providing a normal form for four general points on such a Grassmannian and by explicitly calculating the tangent spaces at these four points.

Contents

1	Introduction	1
2	Known cases in small dimension	3
3	Notations and definitions	3
4	Normal forms	5
5	Embedding and parametrisation	6
6	Tangent space calculation	7
7	Third secant variety	11
8	Fourth secant variety	13
9	Computational experiments	16

1 Introduction

Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety. The r -th secant variety $\sigma_r(X)$ is defined to be the closure of the union of linear spans of all the r -tuples of points lying on X .

It is a long standing and well established problem to calculate properties of secants of certain varieties, in particular homogeneous spaces in their homogeneous embeddings. The dimension is the simplest among those investigated properties, yet even for the easiest homogeneous spaces, the calculation of dimension is a highly

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non-trivial problem, and there is an extensive related literature. A well known classification of defective secants to Veronese embeddings of \mathbb{P}^n was completed in a series of papers by Alexander and Hirschowitz [AH95]. There are corresponding conjecturally complete lists of defective secants to Segre products $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ [AOP09a] and to ordinary Grassmannians $\mathbb{G}(k, n)$ (see [AOP09b], [CGG05] and [BDdG07]), whereas for Segre-Veronese varieties even such a conjectural classification is missing (see [AB09] and numerous references therein).

In this paper we undertake the study of dimensions of secant varieties of Lagrangian Grassmannians $\mathbb{LG}(n, 2n)$ in their minimal homogeneous embeddings. These are projective varieties parametrising dimension n isotropic subspaces of a symplectic vector space V of dimension $2n$.

Theorem 1.1. *Suppose $n \geq 4$, and $r = 3$ or $r = 4$. Then:*

- *If $n = 4$, $r = 3$, then $\dim \sigma_3(\mathbb{LG}(4, 8)) = 31 = (3 * 11 - 1) - 1$.*
- *If $n = 4$, $r = 4$, then $\dim \sigma_4(\mathbb{LG}(4, 8)) = 39 = (4 * 11 - 1) - 4$.*
- *If $n \geq 5$, then $\sigma_3(\mathbb{LG}(n, 2n))$ and $\sigma_4(\mathbb{LG}(n, 2n))$ always have the expected dimension, namely $r(d + 1) - 1$, where $d = \dim \mathbb{LG}(n, 2n) = \binom{n+1}{2}$*

The cases $n \leq 3$ or $r = 2$ are also explained in our paper, but these were known before, see Section 2.

The proof of the theorem is split into the case $r = 3$ and $r = 4$ and explained in Sections 7 and 8, respectively. The idea is to calculate a normal form for four general points of a Lagrangian Grassmannian (see Proposition 4.1) and then perform an explicit calculation of generators of the affine tangent spaces at this normalised general points. By application of Terracini Lemma (see Lemma 3.2 below) the dimension of the secant variety is determined by the dimension of the sum of those affine tangent spaces. Thus the theorem boils down to the calculation of a rank of a certain matrix, whose rows are the generators of the tangent spaces.

In §9 we also present the results of some computational experiments in Magma [BCP97] in small dimensions. As a conclusion we dare to conjecture:

Conjecture 1.2. *The secant variety $\sigma_r(\mathbb{LG}(n, 2n))$ has the expected dimension except for the case $n = 4$ and $r = 3$ (defect 1) and $r = 4$ (defect 4).*

As a corollary of Theorem 1.1 and Proposition 9.1 we know the conjecture is true both for $r \leq 4$ and also for $n \leq 8$.

We point out that Conjecture 1.2 is somehow “expected” from the related conjectural classification of defective secants to ordinary Grassmannians. We quote from [BDdG07, Conjecture 4.1], yet such a conjecture was long before believed to be true.

Conjecture 1.3. *Let $k \geq 3$. Then $\sigma_r(\mathbb{G}(k, n))$ has the expected dimension except for the cases $\sigma_3(\mathbb{G}(3, 7))$ (defect 1), $\sigma_3(\mathbb{G}(4, 8))$ and $\sigma_4(\mathbb{G}(4, 8))$ (defect 1 and 4 respectively) and $\sigma_4(\mathbb{G}(3, 9))$ (defect 2).*

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2 Known cases in small dimension

We briefly review what is known in low dimension.

The first Lagrangian Grassmannian $\mathbb{L}\mathbb{G}(2, 4)$ has dimension $d = 3$ and is a quadric hypersurface in \mathbb{P}^4 , so $\sigma_2(\mathbb{L}\mathbb{G}(2, 4))$ fills the ambient \mathbb{P}^4 .

The variety $\mathbb{L}\mathbb{G}(3, 6)$ has dimension $d = 6$ and

$$\mathbb{L}\mathbb{G}(3, 6) \subset \mathbb{P}^{13} = \mathbb{P} \left(\bigoplus_{i=0}^3 (S^2 \wedge^i \mathbb{C}^3) \right).$$

Again the secant variety $\sigma_2(\mathbb{L}\mathbb{G}(3, 6)) = \mathbb{P}^{13}$ fills the ambient space, as follows from Lemma 6.3 below. This statement was known before — for instance it is contained in the proof of [LM07, Prop. 17(2)], since $\mathbb{L}\mathbb{G}(3, 6)$ is a Legendrian subvariety in \mathbb{P}^{13} .

In general, $\sigma_2(\mathbb{L}\mathbb{G}(n, 2n))$ has the expected dimension for all n , as observed in Lemma 6.3. This must have been known before, although we are unable to find an explicit reference in the literature.

3 Notations and definitions

Throughout the paper we work over an algebraically closed base field of characteristics zero.

If p_1, \dots, p_r are points in \mathbb{P}^N we let $\langle p_1, \dots, p_r \rangle$ denote their linear span. If $X \subset \mathbb{P}^N$, then by definition the r -th secant variety $\sigma_r(X)$ is:

$$\sigma_r(X) = \overline{\bigcup_{p_1, \dots, p_r \in X} \langle p_1, \dots, p_r \rangle}.$$

If $X \subset \mathbb{P}^N$ is non-degenerate and $\dim X = d$, then the dimension of $\sigma_r(X)$ cannot exceed $\min\{N, r(d+1) - 1\}$.

We use a standard terminology, which is summarized in the following definition:

Definition 3.1. Let $X \subset \mathbb{P}^N$ be a non-degenerate variety of dimension d .

1. If $\dim \sigma_r(X) = \min\{N, r(d+1) - 1\}$ we say that $\sigma_r(X)$ *has the expected dimension*.
2. If $\dim \sigma_r(X) < \min\{N, r(d+1) - 1\}$ we say that X is *r -defective*, or that it *has a defective r -th secant variety*.
3. If X is r -defective, its *defect* is the difference $r(d+1) - 1 - \dim \sigma_r(X)$.

The main tool used to compute the dimension of secant varieties is the following well-known lemma by Terracini, see [Zak93, Proposition 1.10]:

Lemma 3.2 (Terracini Lemma). *Let p_1, \dots, p_r be general points in X and let z be a general point of $\langle p_1, \dots, p_r \rangle$. Then the affine tangent space to $\sigma_r(X)$ at z is given by*

$$\hat{T}_z \sigma_r(X) = \hat{T}_{p_1} X + \dots + \hat{T}_{p_r} X$$

where $\hat{T}_{p_i} X$ denotes the affine tangent space to X at p_i .

We call $\mathbb{L}\mathbb{G}(n, 2n)$ the Lagrangian Grassmannian of dimension n Lagrangian subspaces of a complex symplectic vector space V of dimension $2n$. It may be identified with the homogeneous space $\mathrm{Sp}(2n)/P(\alpha_n)$ of dimension $d = \binom{n+1}{2}$. Here α_n is the last simple root of the Lie algebra of $\mathrm{Sp}(2n)$, and $P(\alpha_n) \leq \mathrm{Sp}(2n)$ is the parabolic subgroup obtained by removing the root α_n . (We use the ordering of the roots as in [Bou68].)

We fix a symplectic basis of V , by which we mean that the matrix of the symplectic form in this basis is $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. An element $p \in \mathbb{L}\mathbb{G}(n, 2n)$ is a vector subspace of V , and it can be represented by its basis identified with a $2n \times n$ matrix of rank n that looks like this:

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

with $n \times n$ square matrices B_1 and B_2 . We will say $p = \theta(B)$. Note $p = \theta(B)$ is an isotropic subspace if and only if $B^T J B = 0$, that is if and only if $B_1^T B_2 = B_2^T B_1$.

A presentation of a vector space with a basis clearly requires a choice, so there is a $GL(n)$ -action on all possible choices of B that define the same $p \in \mathbb{L}\mathbb{G}(n, 2n)$:

$$B \cdot g = \begin{bmatrix} B_1 g \\ B_2 g \end{bmatrix}$$

If we restrict our attention to an open affine neighborhood of $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$, consisting of those $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ for which B_1 is invertible, then every point of $\mathbb{L}\mathbb{G}(n, 2n)$ in this neighborhood is represented uniquely by a matrix of the form $\begin{bmatrix} I_n \\ A \end{bmatrix}$ where A is a symmetric $n \times n$ matrix and $A = B_2 B_1^{-1}$. Throughout the paper we use the convention that A has the form

$$\begin{bmatrix} 2a_{11} & a_{12} & \dots & & \\ a_{12} & 2a_{22} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 2a_{nn} \end{bmatrix}.$$

Here a_{ij} are going to be treated as local coordinates on $\mathbb{L}\mathbb{G}(n, 2n)$ around $\theta\left(\begin{bmatrix} I_n \\ 0 \end{bmatrix}\right)$.

4 Normal forms

The Lagrangian Grassmannian $\mathbb{L}\mathbb{G}(n, 2n)$ is a homogeneous space of dimension $d = \binom{n+1}{2}$ with the transitive action of $\mathrm{Sp}(2n)$, a group of dimension $n(2n+1) = 4d - n$. Thus we expect that the quadruples of general points of $\mathbb{L}\mathbb{G}(n, 2n)$ up to the action of $\mathrm{Sp}(2n)$ are parametrised by an n -dimensional family. This is the case and we explicitly describe this family in the following proposition.

Proposition 4.1. *Let $p_1, p_2, p_3, p_4 \in \mathbb{L}\mathbb{G}(n, 2n)$ be four general points. Then there exists a choice of symplectic coordinates on V , such that:*

$$p_1 = \theta \left(\begin{bmatrix} I_n \\ 0 \end{bmatrix} \right) \quad p_2 = \theta \left(\begin{bmatrix} 0 \\ I_n \end{bmatrix} \right) \quad p_3 = \theta \left(\begin{bmatrix} I_n \\ I_n \end{bmatrix} \right) \quad p_4 = \theta \left(\begin{bmatrix} I_n \\ Q_n \end{bmatrix} \right)$$

where $Q_n = \mathrm{diag}(q_1, \dots, q_n)$ is a general $n \times n$ diagonal matrix.

Proof. By homogeneity of $\mathbb{L}\mathbb{G}(n, 2n)$, the choice of the first point is arbitrary. General Lagrangian subspaces are pairwise disjoint. For two disjoint Lagrangian subspaces p_1, p_2 , their direct sum is $p_1 \oplus p_2 = V$ by dimension count. Also the symplectic form identifies p_2 with p_1^* in such a way that the symplectic form on V is the standard symplectic form on $p_1 \oplus p_1^*$. Choose any basis of p_1 and the dual basis of $p_1^* \simeq p_2$ and this gives the normal form of p_2 .

Having fixed p_1 and p_2 , we still have a large subgroup of $\mathrm{Sp}(2n)$ acting on $\mathbb{L}\mathbb{G}(n, 2n)$ and preserving p_1 and p_2 . Namely, this is $GL(n)$ acting as follows. For $g \in GL(n)$, the following matrix $\begin{bmatrix} g^{-1} & 0 \\ 0 & g^T \end{bmatrix}$ is the symplectomorphism representing this action. If $p_3 \in \mathbb{L}\mathbb{G}(n, 2n)$ is a general element, then we may assume it is in the open affine neighborhood of p_1 and thus it is of the form $\theta \left(\begin{bmatrix} I_n \\ A \end{bmatrix} \right)$ for some symmetric matrix A . By generality we may also assume A is non-degenerate. Now:

$$\theta \left(\begin{bmatrix} g^{-1} & 0 \\ 0 & g^T \end{bmatrix} \begin{bmatrix} I_n \\ A \end{bmatrix} \right) = \theta \left(\begin{bmatrix} g^{-1} \\ g^T A \end{bmatrix} \right) = \theta \left(\begin{bmatrix} I_n \\ g^T A g \end{bmatrix} \right).$$

Thus choosing suitable g we may assume $A = I_n$ and we have the normal form for p_3 .

Note that if g is an orthogonal matrix $g^T g = I_n$, then the action of $\begin{bmatrix} g^{-1} & 0 \\ 0 & g^T \end{bmatrix}$ preserves p_1, p_2 and p_3 . Thus it remains to prove that for A a general symmetric matrix, there exists an orthogonal matrix g such that $g^T A g$ is diagonal, or in other words that two general quadratic polynomials can be simultaneously diagonalised. This is a standard fact, see for instance [Rei72, Prop. 2.1(a)&(d)]. \square

5 Embedding and parametrisation

There is a canonical morphism $\bigwedge^{n-2}V \xrightarrow{\wedge\omega} \bigwedge^n V$ (taking into account that $\omega \in \bigwedge^2 V^*$ determines a natural isomorphism between V and V^*). In [Bou59, Chapter 9, §5, n.3] it is shown that this morphism is injective, and that there exists a canonical direct summand of the image which is exactly the weight space V_{ω_n} . (Here V_{ω_n} denotes the irreducible representation of $\mathrm{Sp}(2n)$ with highest weight ω_n , the fundamental weight associated to the last simple root α_n , see [Bou68] for details). In other words we have a splitting:

$$\bigwedge^n V = V_{\omega_n} \oplus \bigwedge^{n-2} V.$$

The image of the Plücker embedding of the Lagrangian Grassmannian $\mathbb{L}\mathbb{G}(n, 2n)$ spans $\mathbb{P}(V_{\omega_n})$ and we have the diagram:

$$\begin{array}{ccc} & & \mathbb{P}(\bigwedge^n \mathbb{C}^{2n}) \\ & \nearrow & \uparrow \\ \mathbb{L}\mathbb{G}(n, 2n) & \xrightarrow{\psi} & \mathbb{P}(V_{\omega_n}) \end{array}$$

In the neighborhood of $\theta \left(\begin{bmatrix} I_n \\ 0 \end{bmatrix} \right)$ the embedding $\psi: \mathbb{L}\mathbb{G}(n, 2n) \hookrightarrow \mathbb{P}(\bigwedge^n \mathbb{C}^{2n})$ is given by sending $\theta \left(\begin{bmatrix} I_n \\ A \end{bmatrix} \right)$ to all the possible $k \times k$ minors of the $n \times n$ symmetric matrix A :

$$\begin{aligned} \psi: \mathbb{L}\mathbb{G}(n, 2n) &\hookrightarrow \mathbb{P}(V_{\omega_n}) \subset \mathbb{P}(\bigwedge^n \mathbb{C}^{2n}) \\ \theta \left(\begin{bmatrix} I_n \\ A \end{bmatrix} \right) &\mapsto [A_{IJ}] \end{aligned} \tag{5.1}$$

with $I, J \subset \{1, \dots, n\}$ and $|I| = |J| = k$, with the convention that the 0×0 minor is just equal to 1. Notice also that $A_{IJ} = A_{JI}$.

It is useful to order the coordinates of $\mathbb{P}(\bigwedge^n \mathbb{C}^{2n})$ in an increasing order, so the index k runs from 0 to n :

$$[1, A, \bigwedge^2 A, \dots, \bigwedge^{n-1} A, \det A].$$

In this order the analogous neighborhood of $\theta \left(\begin{bmatrix} 0 \\ I_n \end{bmatrix} \right)$ consisting of the classes of points $\theta \left(\begin{bmatrix} A \\ I_n \end{bmatrix} \right)$ is described in a symmetric way:

$$[\det A, \bigwedge^{n-1} A^T, \dots, A^T, 1]$$

with the appropriate choices of order and signs of minors. To see this, consider a point which is in both neighborhoods $\theta \left(\begin{bmatrix} I_n \\ A \end{bmatrix} \right) = \theta \left(\begin{bmatrix} A^{-1} \\ I_n \end{bmatrix} \right)$ with A invertible.

Then this point is mapped to:

$$\begin{aligned}
[1, A, \wedge^2 A, \dots, \wedge^{n-1} A, \det A] &= \\
&= \left[\frac{1}{\det A}, \frac{1}{\det A} A, \frac{1}{\det A} \wedge^2 A, \dots, \frac{1}{\det A} \wedge^{n-1} A, 1 \right] \\
&= \left[\det(A^T)^{-1}, \wedge^{n-1}((A^T)^{-1}), \wedge^{n-2}((A^T)^{-1}), \dots, (A^T)^{-1}, 1 \right].
\end{aligned}$$

Here the equality $\frac{1}{\det A} \wedge^k A = \wedge^{n-k}((A^T)^{-1})$ is standard and well known, but rarely explicitly written down. See [Buc09, Prop. H.19], where A is assumed to have determinant one (in the proof sketched there one can easily take into account an arbitrary determinant).

In $\wedge^n V = \wedge^n(W \oplus W^*)$ we distinguish the symmetric part:

$$\begin{aligned}
\wedge^n(W \oplus W^*) &= \bigoplus_{i=0}^n (\wedge^i W \otimes \wedge^{n-i} W^*) \\
&= \bigoplus_{i=0}^n (\wedge^i W \otimes \wedge^i W) \otimes \wedge^n W \\
&= \bigoplus_{i=0}^n (S^2(\wedge^i W) \oplus \wedge^2(\wedge^i W)) \otimes \wedge^n W.
\end{aligned}$$

Since we are interested in the projectivisation of this vector space, the twist by $\wedge^n W$ becomes irrelevant and we regularly skip it. The space V_{ω_n} is always contained in the symmetric part $\bigoplus_{i=0}^n S^2(\wedge^i W)$, however (for $n \geq 4$) it is strictly smaller.

For simpler notation, we will consider all the symmetric minors, rather than its subset. By this we mean that rather than working with the embedding (5.1):

$$\psi : \mathbb{L}\mathbb{G}(n, 2n) \hookrightarrow \mathbb{P}(V_{\omega_n}),$$

we work with embedding φ :

$$\varphi : \mathbb{L}\mathbb{G}(n, 2n) \hookrightarrow \mathbb{P}\left(\bigoplus_{i=0}^n S^2(\wedge^i W)\right). \quad (5.2)$$

6 Tangent space calculation

Given $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J| = k$ and given A_{IJ} the corresponding $k \times k$ minor of the $n \times n$ symmetric matrix A , for any chosen value of k , define:

$$A_{IJ}^{ij} := \left[(-1)^{\#(i,I) + \#(j,J)} A_{I \setminus \{i\} J \setminus \{j\}} + (-1)^{\#(i,J) + \#(j,I)} A_{I \setminus \{j\} J \setminus \{i\}} \right],$$

with the convention that $A_{I \setminus \{i\} J \setminus \{j\}} = 0$ whenever $i \notin I$ or $j \notin J$ and where by $\#(i, I)$ we denote the index of i in I , that is $\#\{i' \in I \mid i' \leq i\}$.

Lemma 6.1. *The affine tangent space $\hat{T}_p(\mathbb{LG}(n, 2n))$ to $\mathbb{LG}(n, 2n)$ at the point $p = \theta \left(\begin{bmatrix} I_n \\ A \end{bmatrix} \right)$ as a subspace of $\bigoplus_{i=0}^n \left(S^2(\wedge^i W) \right)$ is generated by the $d+1$ rows of the matrix:*

$$\left[\begin{array}{c|c|c} 1 & A & \wedge^2 A, \wedge^3 A, \dots, \wedge^{n-1} A, \det A \\ \hline 0 & & \\ \vdots & I_d & A_{IJ}^{ij} \\ 0 & & \end{array} \right]$$

The rows of this matrix are indexed by unordered pairs (i, j) , with $1 \leq i, j \leq n$, and one extra row on the very top of the matrix. The columns are indexed by unordered pairs I, J of subsets of $\{1, \dots, n\}$, with $|I| = |J| \in \{2, \dots, n\}$.

Proof. The affine tangent space $\hat{T}_p \mathbb{LG}(n, 2n)$ at the point $p = \theta \left(\begin{bmatrix} I_n \\ A \end{bmatrix} \right)$ as a subspace of $\bigoplus_{i=0}^n \left(S^2(\wedge^i W) \right)$ is generated by $\varphi(p)$ and the partial derivatives $(\frac{\partial}{\partial a_{ij}} \varphi)|_p$. We have:

$$\begin{aligned} \left(\frac{\partial}{\partial a_{ii}} \varphi|_p \right)_{II} &= \begin{cases} 0 & i \notin I \\ \star & i \in I \end{cases} \\ \left(\frac{\partial}{\partial a_{ii}} \varphi|_p \right)_{IJ} &= \begin{cases} 0 & i \notin I \\ 0 & i \notin J \\ \star & i \in I \cap J \end{cases} \\ \left(\frac{\partial}{\partial a_{ij}} \varphi|_p \right)_{II} &= \begin{cases} 0 & i \notin I \\ 0 & j \notin I \\ \star & i, j \in I \end{cases} \\ \left(\frac{\partial}{\partial a_{ij}} \varphi|_p \right)_{IJ} &= \begin{cases} 0 & i \notin I \\ 0 & j \notin J \\ \star & i \in I \text{ and } j \in J \end{cases} \end{aligned}$$

where the symbol \star is just a placeholder for a non-zero derivative. By expanding the determinant either by row or column we explicitly compute the derivatives \star :

$$\left(\frac{\partial}{\partial a_{ij}} \varphi|_p \right)_{IJ} = A_{IJ}^{ij}.$$

□

Lemma 6.2. *In the notation of Lemma 6.1, if the symmetric matrix A is a diagonal matrix then $A_{IJ}^{ij} = 0$, unless $I = K \cup \{i\}$ and $J = K \cup \{j\}$, for some subset $K \subset \{1, \dots, n\}$.*

Proof. It is an immediate application of the above computations. □

Lemma 6.3. *In the proposed coordinates, for the points p_1 and p_2 as in Proposition 4.1 we have the following equalities for affine tangent spaces as subspaces of $\bigoplus_{i=0}^n (S^2(\bigwedge^i W))$:*

$$\begin{aligned}\hat{T}_{p_1} \mathbb{L}\mathbb{G}(n, 2n) &= S^2(\bigwedge^0 W) \oplus S^2(\bigwedge^1 W) \simeq \mathbb{C} \oplus S^2 W \\ \hat{T}_{p_2} \mathbb{L}\mathbb{G}(n, 2n) &= S^2(\bigwedge^{n-1} W) \oplus S^2(\bigwedge^n W) \simeq (S^2 W^* \oplus \mathbb{C}) \otimes (\bigwedge^n W)^{\otimes 2}\end{aligned}$$

In particular, by Terracini Lemma 3.2 the second secant variety $\sigma_2(\mathbb{L}\mathbb{G}(n, 2n))$ always has the expected dimension.

Proof. Immediate from the given parametrisations around the points p_1 and p_2 . \square

Slightly more demanding is the computation of the tangent space at the points p_3 and p_4 , still from Proposition 4.1.

In the proposition below, for fixed k we divide the $k \times k$ minors of A into 3 groups. First the “on-diagonal” ones, namely those of the form A_{II} . Then “slightly off-diagonal”, namely those of the form A_{IJ} , where $I = K \cup \{l\}$ and $J = K \cup \{m\}$, for some subset $K \subset \{1, \dots, n\}$, with $l, m \in \{1, \dots, n\} \setminus K$ and $l \neq m$. Finally, all the other minors, which in our setup (the choice of points in their normal form), are irrelevant.

Proposition 6.4. *Let p_1, p_2, p_3 and p_4 be four general points of $\mathbb{L}\mathbb{G}(n, 2n)$ in their normal forms as in Proposition 4.1. Then the space*

$$\hat{T}_{p_1}(\mathbb{L}\mathbb{G}(n, 2n)) + \hat{T}_{p_2}(\mathbb{L}\mathbb{G}(n, 2n)) + \hat{T}_{p_3}(\mathbb{L}\mathbb{G}(n, 2n)) + \hat{T}_{p_4}(\mathbb{L}\mathbb{G}(n, 2n))$$

is spanned by the rows of the matrix:

1	0...0	0...0	0...0	0...0	0
$\underline{0}$	I_d	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
0	0...0	0...0	0...0	0...0	1
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	I_d	$\underline{0}$
1	I_n	$\bigwedge^2 I_n$	$\bigwedge^{n-2} I_n$	$\bigwedge^{n-1} I_n$	1
*	*	M_2	M_{n-2}	*	*
1	Q_n	$\bigwedge^2 Q_n$	$\bigwedge^{n-2} Q_n$	$\bigwedge^{n-1} Q_n$	$\det Q_n$
*	*	N_2	N_{n-2}	*	*

(6.5)

where the $*$ are some matrices of the appropriate size, and the matrices M_k and N_k consist of the following blocks:

$$M_k := \begin{array}{c|cc|c} & A_{II} & A_{K \cup \{\ell\} K \cup \{m\}} & A_{IJ} \\ \hline \frac{\partial}{\partial a_{ii}}|_{p_3} & \delta_I^i & \underline{0} & \underline{0} \\ \hline \frac{\partial}{\partial a_{ij}}|_{p_3} & \underline{0} & (-1)^\epsilon \delta_{i\ell} \delta_{jm} & \underline{0} \end{array} \quad (6.6)$$

$$N_k := \begin{array}{c|cc|c} & A_{II} & A_{K \cup \{\ell\} K \cup \{m\}} & A_{IJ} \\ \hline \frac{\partial}{\partial a_{ii}}|_{p_4} & \prod_{j \in I \setminus \{i\}} q_j \delta_I^i & \underline{0} & \underline{0} \\ \hline \frac{\partial}{\partial a_{ij}}|_{p_4} & \underline{0} & (-1)^\epsilon \delta_{i\ell} \delta_{jm} (\prod_{\beta \in K} q_\beta) & \underline{0} \end{array} \quad (6.7)$$

The index k ranges from 2 to $n-2$, $|I| = |J| = k$, $|I \cap J| \leq k-2$. The symbol δ_{ij} is 1 if $i = j$ and 0 otherwise. The symbol δ_I^i is 1 if $i \in I$ and 0 otherwise. Moreover by symmetry we can assume $\ell < m$ and $i < j$. Here $\epsilon = \#(i, K \cup \{\ell\}) + \#(j, K \cup \{m\})$.

Proof. The proposition follows as a corollary of Lemmas 6.1, 6.2 and 6.3.

From Lemma 6.3 it is clear that the first $2d+2$ rows of the stacked Jacobian matrix will have the form above. We also conclude that from having our first two points p_1 and p_2 in their normal forms, we do not care about the first and last columns, in the lower part of the stacked matrix, hence the notation $*$.

Now let us look at the lower part of the stacked Jacobian matrix. We have that $p_3 = \theta \left(\begin{bmatrix} I_n \\ I_n \end{bmatrix} \right)$ and $p_4 = \theta \left(\begin{bmatrix} I_n \\ Q_n \end{bmatrix} \right)$ with a diagonal matrix Q_n , so we can apply Lemma 6.2 to both these points. If the matrix A is diagonal then in particular the minors A_{IJ}^{ij} are nonzero if and only if either $I = J$, $i = j$ and $i \in I$ or else $I = K \cup \{i\}$ and $J = K \cup \{j\}$. The explicit application of Lemma 6.1 for $A = I_n$ and $A = Q_n = \text{diag}(q_{11}, \dots, q_{nn})$ concludes the proof. \square

Carefully looking at matrices (6.5), (6.6) and (6.7) we conclude that in order to calculate the rank of (6.5) (equivalently, the dimension of the fourth secant variety) the following holds:

1. we may use the first $2(d+1)$ rows to eliminate the $*$ parts of the lower rows, and thus the rank in question is equal to $2(d+1)$ plus the rank of the submatrix of the matrix (6.5) obtained by removing the first $2(d+1)$ rows and the first and last $(d+1)$ columns. We call this submatrix B .

2. The submatrix B is a direct sum of two matrices, B^{diag} , consisting of the rows corresponding to $p_3, \frac{\partial}{\partial a_{ii}}|_{p_3}, p_4, \frac{\partial}{\partial a_{ii}}|_{p_4}$ and the columns corresponding to minors A_{II} , and B^{off} , consisting of the remaining rows and columns. Thus

$$\text{rk } B = \text{rk } B^{diag} + \text{rk } B^{off}.$$

7 Third secant variety

From Lemma 6.3 we know that the second secant variety $\sigma_2(\mathbb{LG}(n, 2n))$ always has the expected dimension. For $n \leq 3$ this second secant fills the ambient space. Thus for the rest of the paper we assume that $n \geq 4$. In this section we calculate dimensions of third secant varieties.

Theorem 7.1. *Suppose $n \geq 4$, and $r = 3$. Then:*

- *If $n = 4$, then $\sigma_3(\mathbb{LG}(4, 8))$ has defect 1.*
- *If $n \geq 5$, then $\sigma_3(\mathbb{LG}(n, 2n))$ has the expected dimension.*

Proof. By Terracini Lemma 3.2 and Proposition 6.4, we need to calculate the rank of the first three blocks of the stacked Jacobian matrix (6.5), i.e., the blocks corresponding to the points p_1, p_2 and p_3 and the respective derivatives (so the first $3d + 3$ rows). The expected rank is the maximal one, $3d + 3$, which is the expected dimension of the affine cone of $\sigma_3(\mathbb{LG}(n, 2n))$.

The first $2d + 2$ rows are linearly independent, so we focus our attention on the third block of $d + 1$ rows, and in particular on the submatrices M_k described in (6.6). We need to show that the rows, restricted to the columns corresponding to $k = 2, \dots, n - 2$ have maximal rank $d + 1$ for $n \geq 5$ and rank $d = 10$ for $n = 4$. In case $n = 4$ the index k can only be equal to 2, whereas for $n \geq 5$, we at least have the blocks $k = 2$ and $k = 3$ at our disposal. We will show that the rank of the block corresponding to $k = 2$ is equal to d , but the unique (up to scale) linear relation between the rows does not extend to the block corresponding to $k = 3$.

We “zoom in” the two blocks M_2 and M_3 of the matrix described in (6.6), together with the row coming from the point p_3 . Recall once again that we are ordering the minors by putting first the on-diagonal A_{II} ones and then the off-diagonal A_{IJ} . We also order the elements of the symmetric matrix A in the same on-diagonal, off-diagonal order, so that the on-diagonal block has n rows, and the off-diagonal block has the other $d - n$ rows.

	$A_{\{\ell m\}\{\ell m\}}$	M_2 $A_{\{\beta \ell\}\{\beta m\}}$	$A_{\{\beta \ell\}\{\gamma m\}}$	M_3 A_{II}	$A_{K \cup \{\ell\} K \cup \{m\}}$ $ K =2$	A_{IJ}
1	1...1	0.....0	0...0	1...1	0.....0	0...0
$\frac{\partial}{\partial a_{ii}}$	$\delta_{\{\ell m\}}^i$	<u>0</u>	<u>0</u>	δ_I^i	<u>0</u>	<u>0</u>
$\frac{\partial}{\partial a_{ij}}$	<u>0</u>	$(-1)^\epsilon \delta_{i\ell} \delta_{jm}$	<u>0</u>	<u>0</u>	$(-1)^\epsilon \delta_{i\ell} \delta_{jm}$	<u>0</u>

(7.2)

In the first place, let us focus only on M_2 . The off-diagonal part is of maximal rank $d - n$ because there is precisely one non-zero entry in every column and every row is non-zero. On the other hand the sum of the rows of the on-diagonal block equals twice the first row, the one corresponding to the point p_3 , so we have a linear relation. We claim that the on-diagonal block of M_2 (without the first row corresponding to p_3) is of maximal rank n . Let's "zoom-in" the block M_2 even more, focusing on the on-diagonal part:

	$A_{\{1\ell\}\{1\ell\}}$ $\ell = 2, \dots, n$	$A_{\{2\ell\}\{2\ell\}}$ $\ell = 3, \dots, n$	\dots
$\frac{\partial}{\partial a_{11}}$	1.....1		
$\frac{\partial}{\partial a_{22}}$		1.....1	
\vdots	I_{n-1}		\ddots
\vdots		I_{n-2}	\ddots
$\frac{\partial}{\partial a_{nn}}$			

(7.3)

Notice there is a copy of I_{n-1} , so the rank of the diagonal block in M_2 is at least $n - 1$. But if it is of rank $n - 1$, then the first row of M_2 (the one corresponding to $\frac{\partial}{\partial a_{11}}$) is the sum of all the other rows. One easily verifies this is not the case for instance on the columns corresponding to $A_{\{2\ell\}\{2\ell\}}$. Thus $\text{rk } M_2 = n$ and this finishes the proof of the $n = 4$ case.

Suppose $n \geq 5$, so that for $k = 3$ we have $k \leq n - 2$. By the above argument, the rank of the matrix (7.2) is at least d . If it is equal to d , then the aforementioned relation between rows of M_2 and the first row holds also for the part of the matrix corresponding to the 3×3 minors. This is not the case for any column corresponding to A_{II} — in such a column there are exactly 3 values of i , where δ_I^i is 1, otherwise δ_I^i is 0. Thus the rank of the matrix (7.2) is $d + 1$ and $\sigma_3(\mathbb{LG}(n, 2n))$ is non-defective. \square

8 Fourth secant variety

Continuing the proof of Theorem 1.1 we focus on the remaining case $r = 4$.

Theorem 8.1. *Suppose $n \geq 4$, and $r = 4$. Then:*

- *If $n = 4$, then $\dim \sigma_4(\mathbb{L}\mathbb{G}(4, 8))$ has defect 4.*
- *If $n \geq 5$, then $\sigma_4(\mathbb{L}\mathbb{G}(n, 2n))$ has the expected dimension.*

Proof. We use the fourth point p_4 in the normal form. This time we need to show that the rank of the whole stacked Jacobian matrix (6.5) is equal to 40 for $n = 4$ and $4d + 4$ for $n \geq 5$.

Similarly to the case of $r = 3$, we focus our attention on the last $2d + 2$ rows, and on the middle columns, i.e. on the M_k 's and N_k 's for $k \in \{2, \dots, n - 2\}$. We first claim that $2(d - n)$ rows corresponding to the off-diagonal $\frac{\partial}{\partial a_{ij}}$ at both points p_3 and p_4 are always linearly independent. Indeed, it is enough to look at the appropriate parts of M_2 and N_2 . Fixing a column $A_{\{\beta\ell\}\{\beta m\}}$ there are precisely 2 non-zero entries, the ones corresponding to $\frac{\partial}{\partial a_{\ell m}}$ on both points. Thus the only possible way that there is a linear relation between the $2(d - n)$ rows is that $\frac{\partial}{\partial a_{\ell m}}|_{p_3}$ is proportional to $\frac{\partial}{\partial a_{\ell m}}|_{p_4}$. We write explicitly the (non-zero columns of) two rows to observe that this is not the case. We ignore the sign, as within a column it is the same sign, thus we may multiply the column by -1 if necessary and this obviously does not change the rank.

$$\begin{array}{c|ccc|ccc} & A_{\{1\ell\}\{1m\}} & \dots & A_{\{\beta\ell\}\{\beta m\}} & \dots & & \\ \hline \frac{\partial}{\partial a_{\ell m}}|_{p_3} & 1 & \dots & 1 & \dots & & \\ \frac{\partial}{\partial a_{\ell m}}|_{p_4} & q_1 & & q_\beta & & & \end{array}$$

Here β runs through $\{1, \dots, n\} \setminus \{\ell, m\}$. Clearly, if ℓ or m is 1, then the first column does not show up, but anyway, since $n \geq 4$, there are at least two possible values of β and as long as q_i are not pairwise equal, then the two rows are independent. Since the q_i are general, indeed the $2(d - n)$ off-diagonal rows are independent.

It remains to look at the $2n$ diagonal rows and the 2 rows corresponding to p_3 and p_4 .

First suppose $n = 4$. Then we are able to write the matrix B^{diag} explicitly:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ q_1 q_2 & q_1 q_3 & q_1 q_4 & q_2 q_3 & q_2 q_4 & q_3 q_4 \\ q_2 & q_3 & q_4 & 0 & 0 & 0 \\ q_1 & 0 & 0 & q_3 & q_4 & 0 \\ 0 & q_1 & 0 & q_2 & 0 & q_4 \\ 0 & 0 & q_1 & 0 & q_2 & q_3 \end{pmatrix}.$$

For the case $n \geq 5$ we want to prove that the rank of the submatrix B^{diag} is exactly $2n$ (let us ignore for a second the 2 rows corresponding to p_3 and p_4 , and simply look at the $2n$ diagonal rows). B^{diag} will consist of a copy of the submatrix (7.3) that we have already described, stacked above a matrix of the form:

$$\begin{array}{c|c|c|c}
 & A_{\{1\ell\}\{1\ell\}} & A_{\{2\ell\}\{2\ell\}} & \dots \\
 & \ell = 2, \dots, n & \ell = 3, \dots, n & \dots \\
\hline
\frac{\partial}{\partial a_{11}} & q_2 \ q_3 \ q_4 \ \dots \ q_n & & \\
\hline
\frac{\partial}{\partial a_{22}} & & q_3 \ q_4 \ \dots \ q_n & \\
\vdots & q_1 I_{n-1} & & \ddots \\
\vdots & & q_2 I_{n-2} & \ddots \\
\hline
\frac{\partial}{\partial a_{nn}} & & &
\end{array} \tag{8.2}$$

* *				* *				* *				...					
1	1	...	1														
q_2	q_3	...	q_n														
1				1	1	...	1										
q_1				q_3	q_4	...	q_n										
	1			1				1	1	...	1						
	q_1			q_2				q_4	q_5	...	q_n						
					1			1								
				q_2				q_3								
				1				1	1			
			q_3				...	q_{n-1}	q_n			
														1
										...			1				
													q_{n-2}			q_n	
			1				1				1		1			1
		q_1			q_2				q_3				q_{n-2}	q_{n-1}		

$$A_{\{1,2\}\{1,2\}}, A_{\{1,3\}\{1,3\}}, A_{\{2,3\}\{2,3\}}, A_{\{2,4\}\{2,4\}}, \dots, A_{\{n-5, n-3\}\{n-5, n-3\}}.$$

14

Altogether these columns have a following lower triangular block form:

1	1				
q_2	q_3				
\vdots	\vdots	1	1		
		q_3	q_4		
		\vdots	\vdots	\ddots	
				\ddots	
				\vdots	
				1	1
				q_{n-4}	q_{n-3}
				\vdots	\vdots
					Υ

where Υ is the following 10×10 matrix:

$\{n-4, n-3\}$	$\{n-4, n-2\}$	$\{n-4, n-1\}$	$\{n-4, n\}$	$\{n-3, n-2\}$	$\{n-3, n-1\}$	$\{n-3, n\}$	$\{n-2, n-1\}$	$\{n-2, n\}$	$\{n-1, n\}$
1	1	1	1	0	0	0	0	0	0
q_{n-3}	q_{n-2}	q_{n-1}	q_n	0	0	0	0	0	0
1	0	0	0	1	1	1	0	0	0
q_{n-4}	0	0	0	q_{n-2}	q_{n-1}	q_n	0	0	0
0	1	0	0	1	0	0	1	1	0
0	q_{n-4}	0	0	q_{n-3}	0	0	q_{n-1}	q_n	0
0	0	1	0	0	1	0	1	0	1
0	0	q_{n-4}	0	0	q_{n-3}	0	q_{n-2}	0	q_n
0	0	0	1	0	0	1	0	1	1
0	0	0	q_{n-4}	0	0	q_{n-3}	0	q_{n-2}	q_{n-1}

Since the q_i 's are general, each of the blocks $\begin{bmatrix} 1 & 1 \\ q_i & q_{i+1} \end{bmatrix}$ has rank 2. Also it can be verified (either using a computer algebra system or by a patient Gaussian elimination) that the rank of Υ is 10. Thus $\text{rk } B^{diag} = 2(n-5) + 10 = 2n$.

So all in all the rank of the stacked matrix $\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}$ is $2(d-n)$ from the off-diagonal part, plus $2n$ from the on-diagonal part, so in total $2(d-n) + 2n = 2d$. So this means that there are 2 linear relations among the $2d+2$ rows. It is easy to see what they are (by analogy to the case $r=3$) and these same relations (nor their linear combination) cannot hold on the block $\begin{bmatrix} M_3 \\ N_3 \end{bmatrix}$, and this concludes the proof. \square

9 Computational experiments

For small values of n the secant varieties $\sigma_r(\mathbb{LG}(n, 2n))$ all have the expected dimension, except for the defective cases covered by Theorem 1.1.

Proposition 9.1. *Suppose $n \leq 8$. Then $\sigma_r(\mathbb{LG}(n, 2n))$ have the expected dimension, unless $n = 4$ and $r \in \{3, 4\}$.*

Proof. The proof uses a naive computer code in Magma [BCP97]. The code generates r random points on $\mathbb{LG}(n, 2n)$ (for a slight improvement of time needed to finish the calculation, the first 4 points are assumed to be in the normal forms of Proposition 4.1). Then it calculates the sum of the affine tangent spaces at these points. By Terracini Lemma 3.2 and semicontinuity (since the rank can only drop at special points), the dimension of the sum is a lower bound for the dimension of $\sigma_r(\mathbb{LG}(n, 2n))$.

For each $n \in \{4, \dots, 8\}$, we start the above experiment with $r = 5$ (Theorem 1.1 covers all the cases with $r \leq 4$), and repeatedly add a new point until the dimension of the sum of tangent spaces is equal to the dimension of the ambient vector space. In all the cases we have obtained the lower bound to be equal to the expected dimension, which is an upper bound. Thus the dimension of the secant variety is the expected one for all these cases. \square

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